# HL Paper 2

A particle can move along a straight line from a point $O$. The velocity $v$, in $\mathrm{ms}^{-1}$, is given by the function $v(t) = 1 - \mathrm{e}^{-\sin t^2}$ where time $t \geq 0$ is measured in seconds.

a. Write down the first two times $t_1,\ t_2 > 0$, when the particle changes direction. [2]

b. (i) Find the time $t < t_2$ when the particle has a maximum velocity. [4]

(ii) Find the time $t < t_2$ when the particle has a minimum velocity.

c. Find the distance travelled by the particle between times $t = t_1$ and $t = t_2$. [2]

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Consider the function $f(x) = x^3 - 3x^2 - 9x + 10$ , $x \in \mathbb{R}$.

a. Find the equation of the straight line passing through the maximum and minimum points of the graph $y = f(x)$ . [4]

b. Show that the point of inflexion of the graph $y = f(x)$ lies on this straight line. [2]

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Let $f(x) = x^4 + 0.2x^3 - 5.8x^2 - x + 4,\ x \in \mathbb{R}$.

The domain of $f$ is now restricted to $[0,\ a]$.

Let $g(x) = 2\sin(x - 1) - 3,\ \ -\frac{\pi}{2} + 1 \leqslant x \leqslant \frac{\pi}{2} + 1$.

a. Find the solutions of $f(x) > 0$. [3]

b. For the curve $y = f(x)$. [5]

(i) Find the coordinates of both local minimum points.

(ii) Find the $x$-coordinates of the points of inflexion.

c.i. Write down the largest value of $a$ for which $f$ has an inverse. Give your answer correct to 3 significant figures. [2]

c.ii. For this value of $a$ sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing clearly the coordinates of the end points of each curve. [2]

c.iii Solve $f^{-1}(x) = 1$. [2]

d.i. Find an expression for $g^{-1}(x)$, stating the domain. [4]

d.ii Solve $(f^{-1} \circ g)(x) < 1$. [4]

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a. Given that $2x^3 - 3x + 1$ can be expressed in the form $Ax\left(x^2+1\right) + Bx + C$, find the values of the constants $A$, $B$ and $C$. [2]

b. Hence find $\int \frac{2x^3-3x+1}{x^2+1}\mathrm{d}x$. [5]

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The graphs of $y = x^2\mathrm{e}^{-x}$ and $y = 1 - 2\sin x$ for $2 \leqslant x \leqslant 7$ intersect at points A and B.

The *x*-coordinates of A and B are $x_\mathrm{A}$ and $x_\mathrm{B}$.

a. Find the value of $x_\mathrm{A}$ and the value of $x_\mathrm{B}$. [2]

b. Find the area enclosed between the two graphs for $x_\mathbf{A} \leqslant x \leqslant x_\mathrm{B}$. [3]

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A particle moves in a straight line, its velocity $v$ $\mathrm{ms}^{-1}$ at time $t$ seconds is given by $v = 9t - 3t^2,\ 0 \le t \le 5$.

At time $t = 0$, the displacement $s$ of the particle from an origin $O$ is 3 m.

a. Find the displacement of the particle when $t = 4$. [3]

b. Sketch a displacement/time graph for the particle, $0 \le t \le 5$, showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values. [5]

c. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that $s$ is continuous for all $t \ge 0$. [3]

Given further that $s = 16.5$ when $t = 7.5$, find the values of $a$ and $b$.

d. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that $s$ is continuous for all $t \ge 0$. [4]

Find the times $t_1$ and $t_2 (0 < t_1 < t_2 < 8)$ when the particle returns to its starting point.

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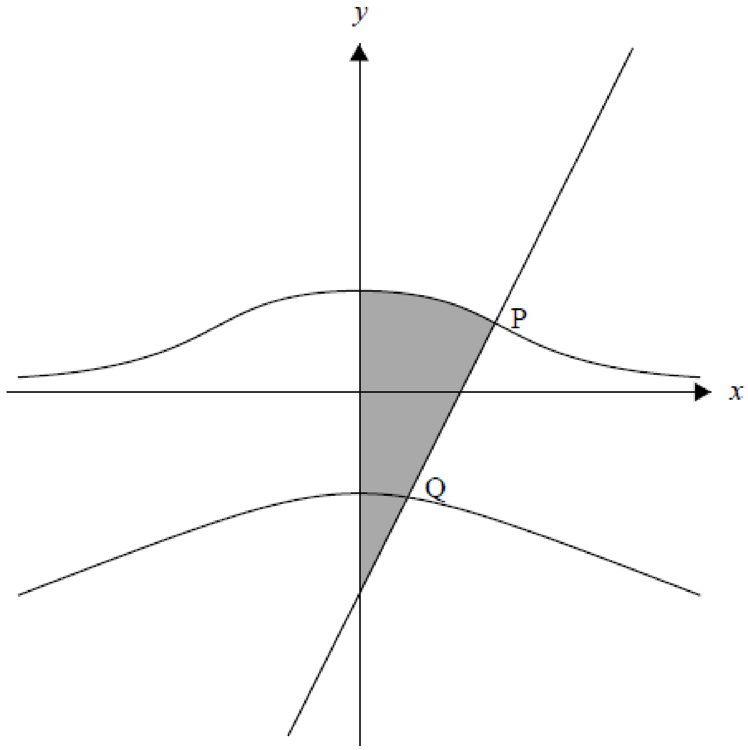
The following graph shows the two parts of the curve defined by the equation $x^2y = 5 - y^4$, and the normal to the curve at the point P(2 , 1).




a. Show that there are exactly two points on the curve where the gradient is zero. [7]

b. Find the equation of the normal to the curve at the point P. [5]

c. The normal at P cuts the curve again at the point Q. Find the $x$-coordinate of Q. [3]

d. The shaded region is rotated by $2\pi$ about the $y$-axis. Find the volume of the solid formed. [7]

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a. Find $\int x\sec^2 x\mathrm{d}x$. [4]

b. Determine the value of $m$ if $\int_0^m x\sec^2 x\mathrm{d}x = 0.5$, where $m > 0$. [2]

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The curve $y = \mathrm{e}^{-x} - x + 1$ intersects the $x$-axis at P.

(a) Find the $x$-coordinate of P.

(b) Find the area of the region completely enclosed by the curve and the coordinate axes.

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The functions $f$ and $g$ are defined by

$$f(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2},\ x \in \mathbb{R}$$

$$g(x) = \frac{e^x - e^{-x}}{2},\ x \in \mathbb{R}$$

Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R},\ n > 1$.

Let $t(x) = \frac{g(x)}{f(x)}$.

a. (i) Show that $\frac{1}{4f(x)-2g(x)} = \frac{e^x}{e^{2x}+3}$. [9]

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x)-2g(x)} dx$. Give your answer in the form $\frac{\pi\sqrt{a}}{b}$ where $a,\ b \in \mathbb{Z}^+$.

b. (i) By forming a quadratic equation in $e^x$, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$. [8]

(ii) Hence or otherwise show that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{n^2 - 1}$ and $k \in \mathbb{R}^+$.

c. (i) Show that $t'(x) = \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$. [6]

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$.

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Find the gradient of the tangent to the curve $x^3y^2 = \cos(\pi y)$ at the point $(-1, 1)$ .

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Consider the function $f(x) = 2\sin^2 x + 7\sin 2x + \tan x - 9,\ 0 \leqslant x < \frac{\pi}{2}$.

Let $u = \tan x$.

a.i. Determine an expression for $f'(x)$ in terms of $x$. [2]

a.ii. Sketch a graph of $y = f'(x)$ for $0 \leqslant x < \frac{\pi}{2}$. [4]

a.iii. Find the $x$-coordinate(s) of the point(s) of inflexion of the graph of $y = f(x)$, labelling these clearly on the graph of $y = f'(x)$. [2]

b.i. Express $\sin x$ in terms of $\mu$. [2]

b.ii. Express $\sin 2x$ in terms of $u$. [3]

b.iii. Hence show that $f(x) = 0$ can be expressed as $u^3 - 7u^2 + 15u - 9 = 0$. [2]

c. Solve the equation $f(x) = 0$, giving your answers in the form $\arctan k$ where $k \in \mathbb{Z}$. [3]

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A function $f$ is defined by $f(x) = \frac{1}{2}(e^x + e^{-x}),\ x \in \mathbb{R}$.

a. (i) Explain why the inverse function $f^{-1}$ does not exist. [14]

(ii) Show that the equation of the normal to the curve at the point P where $x = \ln 3$ is given by $9x + 12y - 9\ln 3 - 20 = 0$.

(iii) Find the $x$-coordinates of the points Q and R on the curve such that the tangents at Q and R pass through $(0, 0)$.

b. The domain of $f$ is now restricted to $x \geqslant 0$. [8]

(i) Find an expression for $f^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y = f(x)$ and the lines $x = 0$ and $y = 5$ is rotated through an angle of $2\pi$ radians about the $y$-axis.

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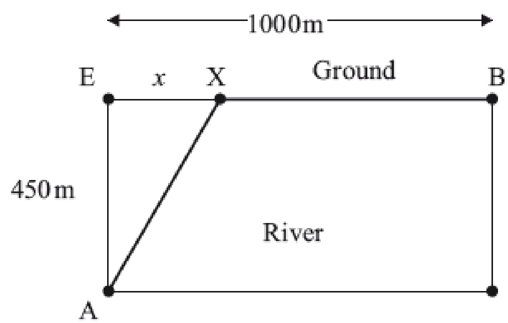
Engineers need to lay pipes to connect two cities A and B that are separated by a river of width 450 metres as shown in the following diagram.

They plan to lay the pipes under the river from A to X and then under the ground from X to B. The cost of laying the pipes under the river is five times the cost of laying the pipes under the ground.

Let $\text{EX} = x$.




Let $k$ be the cost, in dollars per metre, of laying the pipes under the ground.

(a) Show that the total cost $C$, in dollars, of laying the pipes from A to B is given by $C = 5k\sqrt{202\,500 + x^2} + (1000 - x)k$.

(b) (i) Find $\frac{\mathrm{d}C}{\mathrm{d}x}$.

(ii) Hence find the value of $x$ for which the total cost is a minimum, justifying that this value is a minimum.

(c) Find the minimum total cost in terms of $k$.

The angle at which the pipes are joined is $\text{A}\hat{\text{X}}\text{B} = \theta$.

(d) Find $\theta$ for the value of $x$ calculated in (b).

For safety reasons $\theta$ must be at least 120°.

Given this new requirement,

(e) (i) find the new value of $x$ which minimises the total cost;

(ii) find the percentage increase in the minimum total cost.

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The line $y = m(x - m)$ is a tangent to the curve $(1 - x)y = 1$.

Determine $m$ and the coordinates of the point where the tangent meets the curve.

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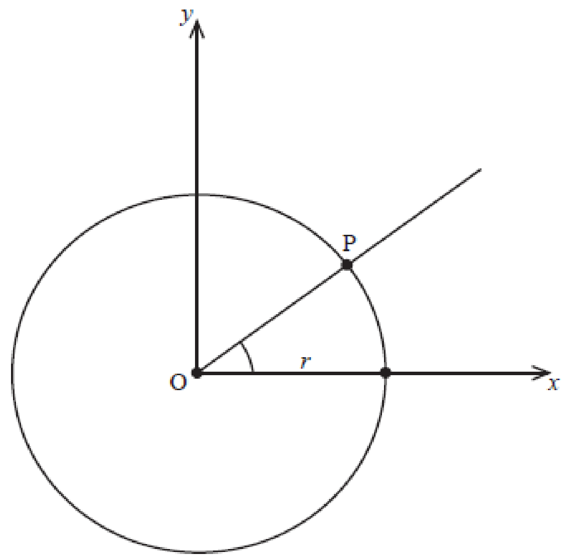
The diagram below shows a circle with centre at the origin O and radius $r > 0$.




A point P($x$ , $y$) , ($x > 0$, $y > 0$) is moving round the circumference of the circle.

Let $m = \tan\left(\arcsin \frac{y}{r}\right)$ .

(a) Given that $\frac{\mathrm{d}y}{\mathrm{d}t} = 0.001r$ , show that $\frac{\mathrm{d}m}{\mathrm{d}t} = \left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$ .

(b) State the geometrical meaning of $\frac{\mathrm{d}m}{\mathrm{d}t}$ .

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The cubic curve $y = 8x^3 + bx^2 + cx + d$ has two distinct points P and Q, where the gradient is zero.

(a) Show that $b^2 > 24c$ .

(b) Given that the coordinates of P and Q are $\left(\frac{1}{2},\ -12\right)$ and $\left(-\frac{3}{2},\ 20\right)$ respectively, find the values of $b$ , $c$ and $d$ .

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Richard, a marine soldier, steps out of a stationary helicopter, 1000 m above the ground, at time $t = 0$. Let his height, in metres, above the ground be given by $s(t)$. For the first 10 seconds his velocity, $v(t)\mathrm{ms}^{-1}$, is given by $v(t) = -10t$.

a. (i) Find his acceleration $a(t)$ for $t < 10$. [6]

(ii) Calculate $v(10)$.

(iii) Show that $s(10) = 500$.

b. At $t = 10$ his parachute opens and his acceleration $a(t)$ is subsequently given by $a(t) = -10 - 5v,\ t \geq 10$. [1]

Given that $\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{1}{\frac{\mathrm{d}v}{\mathrm{d}t}}$, write down $\frac{\mathrm{d}t}{\mathrm{d}v}$ in terms of $v$.

c. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [5]

Hence show that $t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$.

d. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [2]

Hence find an expression for the velocity, $v$, for $t \geq 10$.

e. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [5]

Find an expression for his height, $s$, above the ground for $t \geq 10$.

f. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [2]

Find the value of $t$ when Richard lands on the ground.

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The curve $C$ is defined by equation $xy - \ln y = 1,\ y > 0$.

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$. [4]

b. Determine the equation of the tangent to $C$ at the point $\left(\frac{2}{\mathrm{e}},\ \mathrm{e}\right)$ [3]

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A skydiver jumps from a stationary balloon at a height of 2000 m above the ground.

Her velocity, $v$ $\mathrm{ms}^{-1}$ , $t$ seconds after jumping, is given by $v = 50(1 - \mathrm{e}^{-0.2t})$ .

a. Find her acceleration 10 seconds after jumping. [3]

b. How far above the ground is she 10 seconds after jumping? [3]

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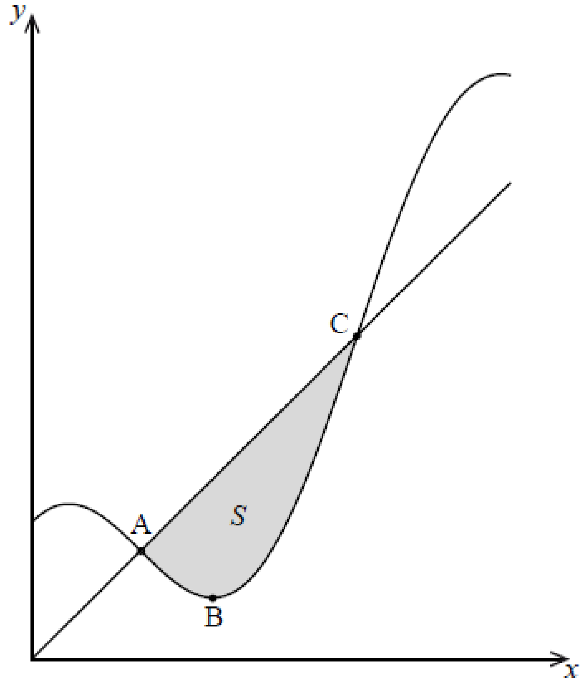
Let $f$ be a function defined by $f(x) = x + 2\cos x$ , $x \in [0,\ 2\pi]$ . The diagram below shows a region $S$ bound by the graph of $f$ and the line $y = x$ .




A and C are the points of intersection of the line $y = x$ and the graph of $f$ , and B is the minimum point of $f$ .

(a) If A, B and C have $x$-coordinates $a\frac{\pi}{2}$, $b\frac{\pi}{6}$ and $c\frac{\pi}{2}$, where $a$ , $b$, $c \in \mathbb{N}$ , find the values of $a$ , $b$ and $c$ .

(b) Find the range of $f$ .

(c) Find the equation of the normal to the graph of f at the point C, giving your answer in the form $y = px + q$ .

(d) The region $S$ is rotated through $2\pi$ about the $x$-axis to generate a solid.

(i) Write down an integral that represents the volume $V$ of this solid.

(ii) Show that $V = 6\pi^2$ .

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A curve *C* is given by the implicit equation $x + y - \cos(xy) = 0$.

The curve $xy = -\frac{\pi}{2}$ intersects *C* at P and Q.

a. Show that $\frac{dy}{dx} = -\left(\frac{1+y\sin(xy)}{1+x\sin(xy)}\right)$. [5]

b.i. Find the coordinates of P and Q. [4]

b.ii. Given that the gradients of the tangents to *C* at P and Q are $m_1$ and $m_2$ respectively, show that $m_1 \times m_2 = 1$. [3]

c. Find the coordinates of the three points on *C*, nearest the origin, where the tangent is parallel to the line $y = -x$. [7]

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Consider the curve with equation $\left(x^2 + y^2\right)^2 = 4xy^2$.

a. Use implicit differentiation to find an expression for $\frac{dy}{dx}$. [5]

b. Find the equation of the normal to the curve at the point (1, 1). [3]

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An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, $2\pi$ radians about the $y$-axis. Units on the coordinate axes are defined to be in centimetres.

a. When the glass contains water to a height $h$ cm, find the volume $V$ of water in terms of $h$. [3]

b. If the water in the glass evaporates at the rate of 3 $cm^3$ per hour for each $cm^2$ of exposed surface area of the water, show that, [6]

$\frac{dV}{dt} = -3\sqrt{2\pi V}$, where $t$ is measured in hours.

c. If the glass is filled completely, how long will it take for all the water to evaporate? [7]

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Xavier, the parachutist, jumps out of a plane at a height of $h$ metres above the ground. After free falling for 10 seconds his parachute opens. His velocity, $v\,\mathrm{ms}^{-1}$, $t$ seconds after jumping from the plane, can be modelled by the function

$$v(t) = \begin{cases} 9.8t, & 0 \leqslant t \leqslant 10 \\ \frac{98}{\sqrt{1+(t-10)^2}}, & t > 10 \end{cases}$$

His velocity when he reaches the ground is $2.8\ \mathrm{ms}^{-1}$.

a. Find his velocity when $t = 15$. [2]

b. Calculate the vertical distance Xavier travelled in the first 10 seconds. [2]

c. Determine the value of $h$. [5]

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The function $f(x) = 3\sin x + 4\cos x$ is defined for $0 < x < 2\pi$.

a. Write down the coordinates of the minimum point on the graph of $f$. [1]

b. The points $\mathrm{P}(p,\ 3)$ and $\mathrm{Q}(q,\ 3)$, $q > p$, lie on the graph of $y = f(x)$. [2]

Find $p$ and $q$.

c. Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3. [4]

d. Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. [7]

Find the equation of the normal to the curve $x^3y^3 - xy = 0$ at the point (1, 1).

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A curve is defined $x^2 - 5xy + y^2 = 7$.

a. Show that $\frac{dy}{dx} = \frac{5y-2x}{2y-5x}$. [3]

b. Find the equation of the normal to the curve at the point $(6,\ 1)$. [4]

c. Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. [8]

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The point P, with coordinates $(p,\ q)$ , lies on the graph of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ , $a > 0$ .

The tangent to the curve at P cuts the axes at $(0, m)$ and $(n, 0)$ . Show that $m + n = a$ .

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Consider the differential equation $y\frac{dy}{dx} = \cos 2x$.

a. (i) Show that the function $y = \cos x + \sin x$ satisfies the differential equation. [10]

(ii) Find the general solution of the differential equation. Express your solution in the form $y = f(x)$, involving a constant of integration.

(iii) For which value of the constant of integration does your solution coincide with the function given in part (i)?

b. A different solution of the differential equation, satisfying $y = 2$ when $x = \frac{\pi}{4}$, defines a curve $C$. [12]

(i) Determine the equation of $C$ in the form $y = g(x)$ , and state the range of the function $g$.

A region $R$ in the $xy$ plane is bounded by $C$, the $x$-axis and the vertical lines $x = 0$ and $x = \frac{\pi}{2}$.

(ii) Find the area of $R$.

(iii) Find the volume generated when that part of $R$ above the line $y = 1$ is rotated about the $x$-axis through $2\pi$ radians.

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By using the substitution $x = 2\tan u$, show that $\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{-\sqrt{x^2+4}}{4x} + C$.

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The function $f$ is defined by $f(x) = x\sqrt{9 - x^2} + 2\arcsin\left(\frac{x}{3}\right)$.

(a) Write down the largest possible domain, for each of the two terms of the function, $f$, and hence state the largest possible domain, $D$ , for $f$.

(b) Find the volume generated when the region bounded by the curve $y = f(x)$, the $x$-axis, the $y$-axis and the line $x = 2.8$ is rotated through $2\pi$ radians about the $x$-axis.

(c) Find $f'(x)$ in simplified form.

(d) **Hence** show that $\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}} dx = 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$, where $p \in D$.

(e) Find the value of $p$ which maximises the value of the integral in (d).

(f) (i) Show that $f''(x) = \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$.

(ii) Hence justify that $f(x)$ has a point of inflexion at $x = 0$, but not at $x = \pm\sqrt{\frac{25}{2}}$.

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A body is moving through a liquid so that its acceleration can be expressed as

$$\left(-\frac{v^2}{200} - 32\right) \text{ms}^{-2},$$

where $v$ ms$^{-1}$ is the velocity of the body at time $t$ seconds.

The initial velocity of the body was known to be 40 ms$^{-1}$.

(a) Show that the time taken, $T$ seconds, for the body to slow to $V$ ms$^{-1}$ is given by

$$T = 200\int_{V}^{40} \frac{1}{v^2 + 80^2} dv.$$

(b) (i) Explain why acceleration can be expressed as $v\frac{dv}{ds}$, where $s$ is displacement, in metres, of the body at time $t$ seconds.

(ii) **Hence** find a similar integral to that shown in part (a) for the distance, $S$ metres, travelled as the body slows to $V$ ms$^{-1}$.

(c) **Hence**, using parts (a) and (b), find the distance travelled and the time taken until the body momentarily comes to rest.

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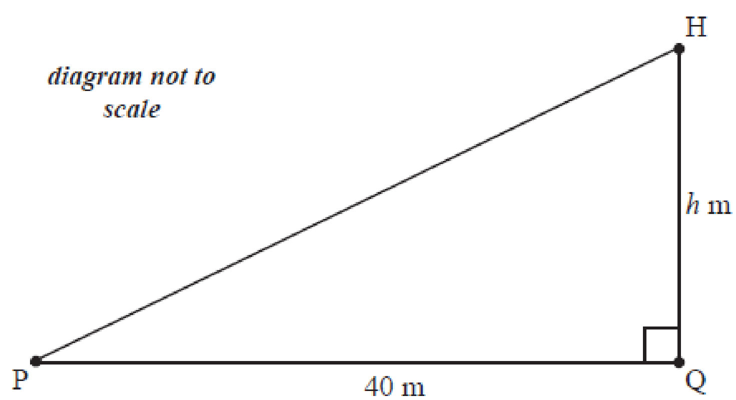
A helicopter H is moving vertically upwards with a speed of 10 ms$^{-1}$. The helicopter is $h$ m directly above the point Q which is situated on level ground. The helicopter is observed from the point P which is also at ground level and PQ $= 40$ m. This information is represented in the diagram below.



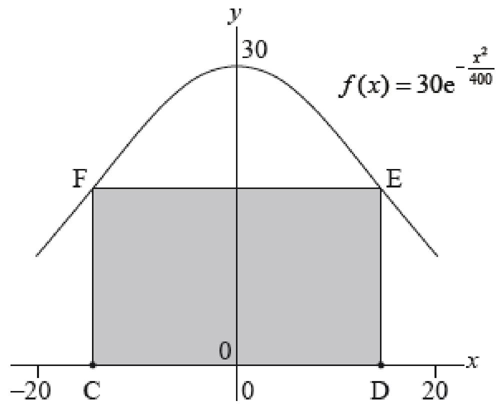

When $h = 30$,

(a) show that the rate of change of $\text{H}\hat{\text{P}}\text{Q}$ is 0.16 radians per second;

(b) find the rate of change of PH.

The following diagram shows a vertical cross section of a building. The cross section of the roof of the building can be modelled by the curve $f(x) = 30\mathrm{e}^{-\frac{x^2}{400}}$, where $-20 \le x \le 20$.

Ground level is represented by the $x$-axis.




a. Find $f''(x)$. [4]

b. Show that the gradient of the roof function is greatest when $x = -\sqrt{200}$. [3]

c. The cross section of the living space under the roof can be modelled by a rectangle $CDEF$ with points $\mathrm{C}(-a,\ 0)$ and $\mathrm{D}(a,\ 0)$, where $0 < a \le 20$. [5]

Show that the maximum area $A$ of the rectangle $CDEF$ is $600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$.

d. A function $I$ is known as the Insulation Factor of $CDEF$. The function is defined as $I(a) = \frac{P(a)}{A(a)}$ where $\mathrm{P} = \mathrm{Perimeter}$ and $\mathrm{A} = \mathrm{Area\ of\ the\ rectangle.}$ [9]

(i) Find an expression for $P$ in terms of $a$.

(ii) Find the value of $a$ which minimizes $I$.

(iii) Using the value of $a$ found in part (ii) calculate the percentage of the cross sectional area under the whole roof that is not included in the cross section of the living space.

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Consider the curve with equation $x^3 + y^3 = 4xy$.

The tangent to this curve is parallel to the $x$-axis at the point where $x = k,\ k > 0$.

a. Use implicit differentiation to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4y-3x^2}{3y^2-4x}$. [3]

b. Find the value of $k$. [5]

Consider the function $f(x) = \frac{\sqrt{x}}{\sin x},\ 0 < x < \pi$.

Consider the region bounded by the curve $y = f(x)$, the $x$-axis and the lines $x = \frac{\pi}{6},\ x = \frac{\pi}{3}$.

a.i. Show that the $x$-coordinate of the minimum point on the curve $y = f(x)$ satisfies the equation $\tan x = 2x$. [5]

a.ii. Determine the values of $x$ for which $f(x)$ is a decreasing function. [2]

b. Sketch the graph of $y = f(x)$ showing clearly the minimum point and any asymptotic behaviour. [3]

c. Find the coordinates of the point on the graph of $f$ where the normal to the graph is parallel to the line $y = -x$. [4]

d. This region is now rotated through $2\pi$ radians about the $x$-axis. Find the volume of revolution. [3]

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The acceleration of a car is $\frac{1}{40}(60 - v)$ $\text{ms}^{-2}$, when its velocity is $v$ $\text{ms}^{-2}$. Given the car starts from rest, find the velocity of the car after 30 seconds.

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A particle moves along a straight line so that after $t$ seconds its displacement $s$ , in metres, satisfies the equation $s^2 + s - 2t = 0$ . Find, in terms of $s$ , expressions for its velocity and its acceleration.

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Consider $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right)$

The function $f$ is defined by $f(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right),\ x \in D$

The function $g$ is defined by $g(x) = -1 + \ln\left(\sqrt{x^2 - 1}\right),\ x \in ]1,\ \infty[$.

a. Find the largest possible domain $D$ for $f$ to be a function. [2]

b. Sketch the graph of $y = f(x)$ showing clearly the equations of asymptotes and the coordinates of any intercepts with the axes. [3]

c. Explain why $f$ is an even function. [1]

d. Explain why the inverse function $f^{-1}$ does not exist. [1]

e. Find the inverse function $g^{-1}$ and state its domain. [4]

f. Find $g'(x)$. [3]

g.i. Hence, show that there are no solutions to $g'(x) = 0$; [2]

g.ii. Hence, show that there are no solutions to $(g^{-1})'(x) = 0$. [2]

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Particle $A$ moves such that its velocity $v$ $\text{ms}^{-1}$, at time $t$ seconds, is given by $v(t) = \frac{t}{12+t^4}$, $t \geqslant 0$.

Particle $B$ moves such that its velocity $v$ $\text{ms}^{-1}$ is related to its displacement $s$ m, by the equation $v(s) = \arcsin\left(\sqrt{s}\right)$.

a. Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. [2]

b. Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4} \mathrm{d}t$. [4]

c. Find the exact distance travelled by particle $A$ between $t = 0$ and $t = 6$ seconds. [3]

Give your answer in the form $k \arctan(b),\ k,\ b \in \mathbb{R}$.

d. Find the acceleration of particle B when $s = 0.1$ m. [3]

---

A function is defined by $f(x) = x^2 + 2,\ x \geq 0$. A region $R$ is enclosed by $y = f(x)$, the $y$-axis and the line $y = 4$.

a. (i) Express the area of the region $R$ as an integral with respect to $y$. [3]

(ii) Determine the area of $R$, giving your answer correct to four significant figures.

b. Find the exact volume generated when the region $R$ is rotated through $2\pi$ radians about the $y$-axis. [3]

---

A function $f$ is defined by $f(x) = x^3 + \mathrm{e}^x + 1,\ x \in \mathbb{R}$. By considering $f'(x)$ determine whether $f$ is a one-to-one or a many-to-one function.

---

(a) Integrate $\int \frac{\sin\theta}{1-\cos\theta} \mathrm{d}\theta$ .

(b) Given that $\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta} \mathrm{d}\theta = \frac{1}{2}$ and $\frac{\pi}{2} < a < \pi$, find the value of $a$ .

---

Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of $k$ .

A ladder of length 10 m on horizontal ground rests against a vertical wall. The bottom of the ladder is moved away from the wall at a constant speed of $0.5\ \text{ms}^{-1}$. Calculate the speed of descent of the top of the ladder when the bottom of the ladder is 4 m away from the wall.

---

Let the function $f$ be defined by $f(x) = \frac{2-e^x}{2e^x-1}$, $x \in D$.

a. Determine $D$, the largest possible domain of $f$. [2]

b. Show that the graph of $f$ has three asymptotes and state their equations. [5]

c. Show that $f'(x) = -\frac{3e^x}{(2e^x-1)^2}$. [3]

d. Use your answers from parts (b) and (c) to justify that $f$ has an inverse and state its domain. [4]

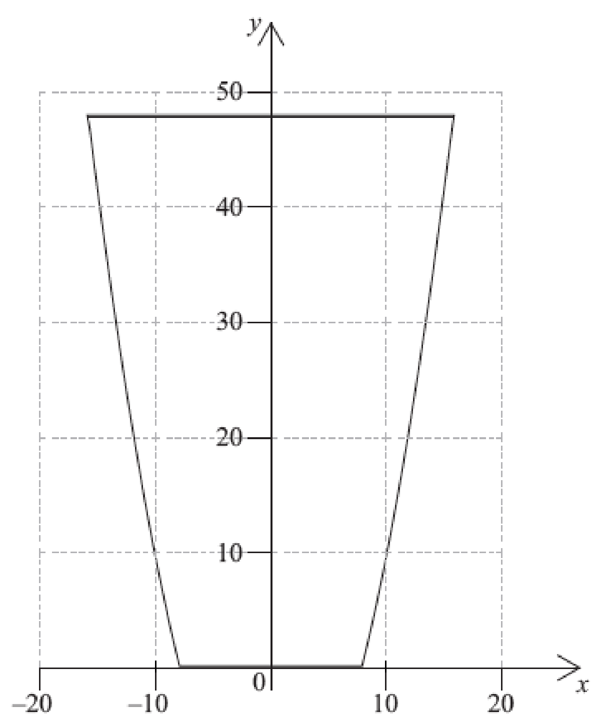
e. Find an expression for $f^{-1}(x)$. [4]

f. Consider the region $R$ enclosed by the graph of $y = f(x)$ and the axes. [4]

Find the volume of the solid obtained when $R$ is rotated through $2\pi$ about the $y$-axis.

---

The vertical cross-section of a container is shown in the following diagram.




The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is $48$ cm.

a. If the container is filled with water to a depth of $h$ cm, show that the volume, $V$ cm$^3$, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. [3]

b. The container, initially full of water, begins leaking from a small hole at a rate given by \(\frac{{{\text{d}}V}}{{{\text{d}}t}} = - \frac{{250\sqrt h }}{{\pi(h + 16)}}\) where $t$ is measured in seconds. [10]

(i) Show that \(\frac{{{\text{d}}h}}{{{\text{d}}t}} = - \frac{{250\sqrt h }}{{4{\pi ^2}{{(h + 16)}^2}}}\).

(ii) State $\frac{\mathrm{d}t}{\mathrm{d}h}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. ($60$ seconds = 1 minute)

c. Once empty, water is pumped back into the container at a rate of $8.5\ \mathrm{cm}^3\mathrm{s}^{-1}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\mathrm{cm}^3\mathrm{s}^{-1}$. [3]

Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container.

---

Let $f(x) = \frac{\mathrm{e}^{2x}+1}{\mathrm{e}^{x}-2}$.

The line $L_2$ is parallel to $L_1$ and tangent to the curve $y = f(x)$.

a. Find the equations of the horizontal and vertical asymptotes of the curve $y = f(x)$. [4]

b. (i) Find $f'(x)$. [8]

(ii) Show that the curve has exactly one point where its tangent is horizontal.

(iii) Find the coordinates of this point.

c. Find the equation of $L_1$, the normal to the curve at the point where it crosses the $y$-axis. [4]

d. Find the equation of the line $L_2$. [5]

---

The region $A$ is enclosed by the graph of $y = 2\arcsin(x-1) - \frac{\pi}{4}$, the $y$-axis and the line $y = \frac{\pi}{4}$.

a. Write down a definite integral to represent the area of $A$. [4]

b. Calculate the area of $A$. [2]

---

The displacement, $s$, in metres, of a particle $t$ seconds after it passes through the origin is given by the expression $s = \ln(2 - e^{-t}),\ t \geqslant 0$.

a. Find an expression for the velocity, $v$, of the particle at time $t$. [2]

b. Find an expression for the acceleration, $a$, of the particle at time $t$. [2]

c. Find the acceleration of the particle at time $t = 0$. [1]

(a) Differentiate $f(x) = \arcsin x + 2\sqrt{1-x^2}$ , $x \in [-1, 1]$ .

(b) Find the coordinates of the point on the graph of $y = f(x)$ in $[-1, 1]$, where the gradient of the tangent to the curve is zero.

---

Find the volume of the solid formed when the region bounded by the graph of $y = \sin(x-1)$, and the lines $y = 0$ and $y = 1$ is rotated by $2\pi$ about the $y$-axis.

---

By using an appropriate substitution find

$$\int \frac{\tan(\ln y)}{y} \mathrm{d}y, \ y > 0 .$$

---

A stalactite has the shape of a circular cone. Its height is 200 mm and is increasing at a rate of 3 mm per century. Its base radius is 40 mm and is decreasing at a rate of 0.5 mm per century. Determine if its volume is increasing or decreasing, and the rate at which the volume is changing.

---

Consider the triangle PQR where $\mathrm{Q\hat{P}R} = 30^\circ$, $\mathrm{PQ} = (x+2)$ cm and $\mathrm{PR} = (5-x)^2$ cm, where $-2 < x < 5$.

a. Show that the area, $A$ $\mathrm{cm}^2$, of the triangle is given by $A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$. [2]

b. (i) State $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(ii) Verify that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$.

c. (i) Find $\frac{\mathrm{d}^2 A}{\mathrm{d}x^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle $PQR$. [7]

(ii) State the maximum area of triangle $PQR$.

(iii) Find $QR$ when the area of triangle $PQR$ is a maximum.

---

The particle $P$ moves along the $x$-axis such that its velocity, $v$ $\mathrm{ms}^{-1}$ , at time $t$ seconds is given by $v = \cos(t^2)$.

a. Given that $P$ is at the origin O at time $t = 0$ , calculate [4]

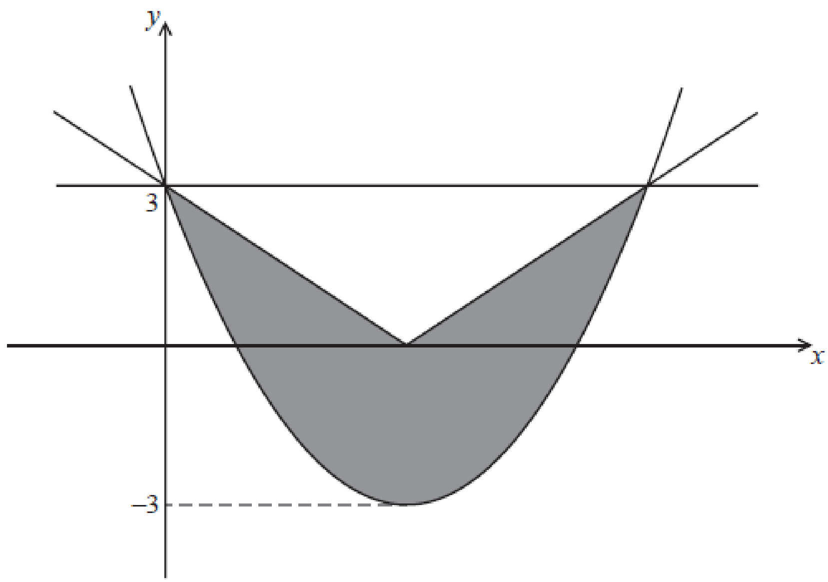
(i) the displacement of $P$ from O after 3 seconds;

(ii) the total distance travelled by $P$ in the first 3 seconds.

b. Find the time at which the total distance travelled by $P$ is 1 m. [2]

---

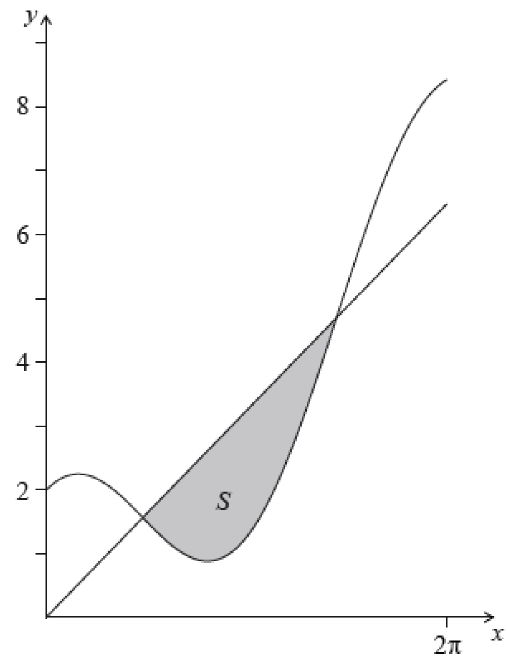
The diagram below shows the graphs of $y = \left|\frac{3}{2}x - 3\right|,\ y = 3$ and a quadratic function, that all intersect in the same two points.




Given that the minimum value of the quadratic function is −3, find an expression for the area of the shaded region in the form $\int_0^t (ax^2 + bx + c)\mathrm{d}x$, where the constants $a$, $b$, $c$ and $t$ are to be determined. (Note: The integral does not need to be evaluated.)

---

The shaded region $S$ is enclosed between the curve $y = x + 2\cos x$, for $0 \leqslant x \leqslant 2\pi$, and the line $y = x$, as shown in the diagram below.




a. Find the coordinates of the points where the line meets the curve. [3]

b. The region $S$ is rotated by $2\pi$ about the $x$-axis to generate a solid. [5]

(i) Write down an integral that represents the volume $V$ of the solid.

(ii) Find the volume $V$.

---

Consider the curve, $C$ defined by the equation $y^2 - 2xy = 5 - e^x$. The point A lies on $C$ and has coordinates $(0,\ a),\ a > 0$.

a. Find the value of $a$. [2]

b. Show that $\frac{dy}{dx} = \frac{2y - e^x}{2(y-x)}$. [4]

c. Find the equation of the normal to $C$ at the point A. [3]

d. Find the coordinates of the second point at which the normal found in part (c) intersects $C$. [4]

e. Given that $v = y^3,\ y > 0$, find $\frac{dv}{dx}$ at $x = 0$. [3]

---

If $y = \ln\left(\frac{1}{3}(1 + e^{-2x})\right)$, show that $\frac{dy}{dx} = \frac{2}{3}(e^{-y} - 3)$ .

---

A point P moves in a straight line with velocity $v$ms$^{-1}$ given by $v(t) = e^{-t} - 8t^2e^{-2t}$ at time $t$ seconds, where $t \geq 0$.

a. Determine the first time $t_1$ at which P has zero velocity. [2]

b.i. Find an expression for the acceleration of P at time $t$. [2]

b.ii. Find the value of the acceleration of P at time $t_1$. [1]

---

Let $f(x) = x(x+2)^6$.

a. Solve the inequality $f(x) > x$. [5]

b. Find $\int f(x)\mathrm{d}x$. [5]

---

Consider the function $f$ , defined by $f(x) = x - a\sqrt{x}$ , where $x \geqslant 0$, $a \in \mathbb{R}^+$ .

(a) Find in terms of $a$

(i) the zeros of $f$ ;

(ii) the values of $x$ for which $f$ is decreasing;

(iii) the values of $x$ for which $f$ is increasing;

(iv) the range of $f$ .

(b) State the concavity of the graph of $f$ .

---

By using the substitution $x^2 = 2\sec\theta$, show that $\int \frac{\mathrm{d}x}{x\sqrt{x^4-4}} = \frac{1}{4}\arccos\left(\frac{2}{x^2}\right) + c$.

---

The graph of $y = \ln(5x+10)$ is obtained from the graph of $y = \ln x$ by a translation of $a$ units in the direction of the $x$-axis followed by a translation of $b$ units in the direction of the $y$-axis.

a. Find the value of $a$ and the value of $b$. [4]

b. The region bounded by the graph of $y = \ln(5x+10)$, the $x$-axis and the lines $x = \mathrm{e}$ and $x = 2\mathrm{e}$, is rotated through $2\pi$ radians about the $x$-axis. Find the volume generated. [2]

---

Two cyclists are at the same road intersection. One cyclist travels north at $20\,\mathrm{km\,h^{-1}}$. The other cyclist travels west at $15\,\mathrm{km\,h^{-1}}$.

Use calculus to show that the rate at which the distance between the two cyclists changes is independent of time.

---

A bicycle inner tube can be considered as a joined up cylinder of fixed length $200$ cm and radius $r$ cm. The radius $r$ increases as the inner tube is pumped up. Air is being pumped into the inner tube so that the volume of air in the tube increases at a constant rate of $30\ \mathrm{cm^3 s^{-1}}$. Find the rate at which the radius of the inner tube is increasing when $r = 2\ \mathrm{cm}$.

---

The region $R$ is enclosed by the graph of $y = e^{-x^2}$, the $x$-axis and the lines $x = -1$ and $x = 1$.

Find the volume of the solid of revolution that is formed when $R$ is rotated through $2\pi$ about the $x$-axis.

---

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

a. Find the equation of the normal to the curve at the point $\left(1,\ \sqrt{3}\right)$. [6]

b. Find the volume of the solid formed when the region bounded by the curve, the $x$-axis for $x \geqslant 0$ and the $y$-axis for $y \geqslant 0$ is rotated through $2\pi$ about the $x$-axis. [3]

---

A. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$, [8]

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

B. (a) Using integration by parts, show that $\int \mathrm{e}^{2x} \sin x \mathrm{d}x = \frac{1}{5}\mathrm{e}^{2x}(2\sin x - \cos x) + C$. [17]

(b) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 - y^2}\mathrm{e}^{2x} \sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form $y = f(x)$.

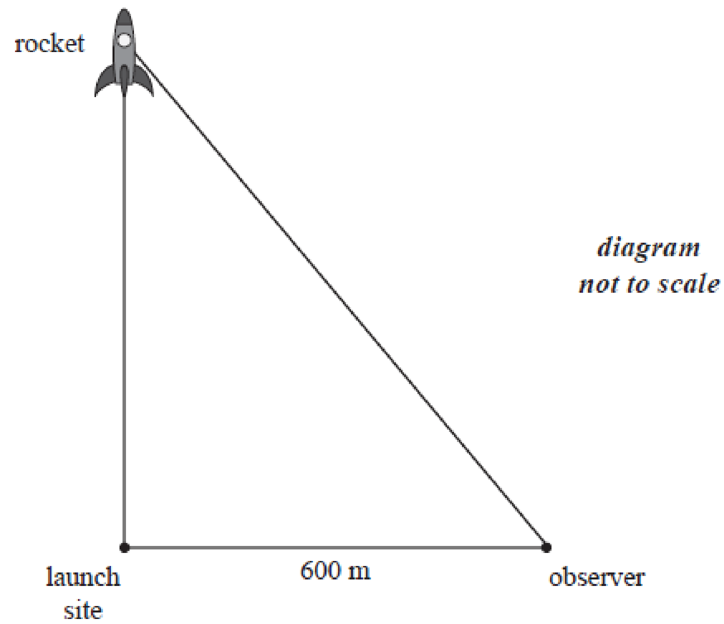
(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the $x$-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the $x$-axis, between the origin and P, is rotated 360° about the $x$-axis to form a solid of revolution.

Calculate the volume of this solid.

A rocket is rising vertically at a speed of 300 $\text{ms}^{-1}$ when it is 800 m directly above the launch site. Calculate the rate of change of the distance between the rocket and an observer, who is 600 m from the launch site and on the same horizontal level as the launch site.




---

A particle moves such that its velocity $v\,\text{ms}^{-1}$ is related to its displacement $s$ m, by the equation $v(s) = \arctan(\sin s),\ 0 \leqslant s \leqslant 1$. The particle's acceleration is $a\,\text{ms}^{-2}$.

a. Find the particle's acceleration in terms of $s$. [4]

b. Using an appropriate sketch graph, find the particle's displacement when its acceleration is $0.25\ \text{ms}^{-2}$. [2]

---

The function $f$ has inverse $f^{-1}$ and derivative $f'(x)$ for all $x \in \mathbb{R}$. For all functions with these properties you are given the result that for $a \in \mathbb{R}$ with $b = f(a)$ and $f'(a) \neq 0$

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

a. Verify that this is true for $f(x) = x^3 + 1$ at $x = 2$. [6]

b. Given that $g(x) = xe^{x^2}$, show that $g'(x) > 0$ for all values of $x$. [3]

c. Using the result given at the start of the question, find the value of the gradient function of $y = g^{-1}(x)$ at $x = 2$. [4]

d. (i) With $f$ and $g$ as defined in parts (a) and (b), solve $g \circ f(x) = 2$. [6]

(ii) Let $h(x) = (g \circ f)^{-1}(x)$. Find $h'(2)$.

A family of cubic functions is defined as $f_k(x) = k^2x^3 - kx^2 + x,\ k \in \mathbb{Z}^+$ .

(a) Express in terms of $k$

(i) $f'_k(x)$ and $f''_k(x)$ ;

(ii) the coordinates of the points of inflexion $P_k$ on the graphs of $f_k$ .

(b) Show that all $P_k$ lie on a straight line and state its equation.

(c) Show that for all values of $k$, the tangents to the graphs of $f_k$ at $P_k$ are parallel, and find the equation of the tangent lines.

---

A triangle is formed by the three lines $y = 10 - 2x,\ y = mx$ and $y = -\frac{1}{m}x$, where $m > \frac{1}{2}$.

Find the value of $m$ for which the area of the triangle is a minimum.

---

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$ , at time $t$ seconds, is given by

$$v(t) = \begin{cases} 5 - (t-2)^2, & 0 \le t \le 4 \\ 3 - \frac{t}{2}, & t > 4 \end{cases}.$$

a. Find the value of $t$ when the particle is instantaneously at rest. [2]

b. The particle returns to its initial position at $t = T$. [5]

Find the value of $T$.

---

Let $f(x) = \frac{a + be^x}{ae^x + b}$, where $0 < b < a$.

(a) Show that $f'(x) = \frac{(b^2 - a^2)e^x}{(ae^x + b)^2}$.

(b) **Hence** justify that the graph of $f$ has no local maxima or minima.

(c) Given that the graph of $f$ has a point of inflexion, find its coordinates.

(d) Show that the graph of $f$ has exactly two asymptotes.

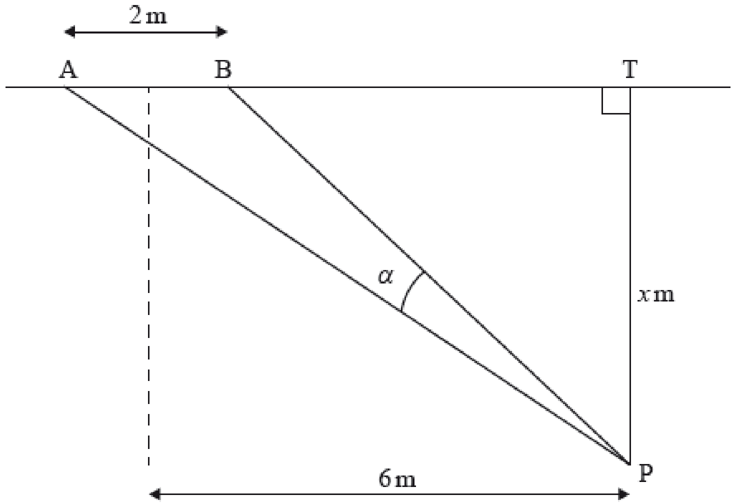
(e) Let $a = 4$ and $b = 1$. Consider the region $R$ enclosed by the graph of $y = f(x)$, the $y$-axis and the line with equation $y = \frac{1}{2}$.

Find the volume $V$ of the solid obtained when $R$ is rotated through $2\pi$ about the $x$-axis.

---

Points A , B and T lie on a line on an indoor soccer field. The goal, [AB] , is 2 metres wide. A player situated at point P kicks a ball at the goal. [PT] is perpendicular to (AB) and is 6 metres from a parallel line through the centre of [AB] . Let PT be $x$ metros and let $\alpha = \mathrm{A\hat{P}B}$ measured in degrees.

Assume that the ball travels along the floor.




The maximum for $\tan\alpha$ gives the maximum for $\alpha$.

a. Find the value of $\alpha$ when $x = 10$. [4]

b. Show that $\tan\alpha = \frac{2x}{x^2+35}$. [4]

c. (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha)$. [11]

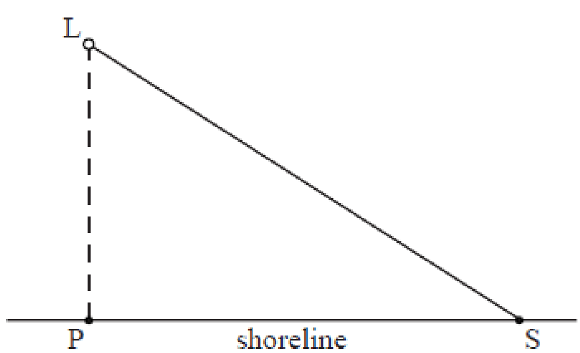
(ii) Hence or otherwise find the value of $\alpha$ such that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0$.

(iii) Find $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ and hence show that the value of $\alpha$ never exceeds 10°.

d. Find the set of values of $x$ for which $\alpha \geqslant 7^\circ$. [3]

---

A lighthouse L is located offshore, 500 metres from the nearest point P on a long straight shoreline. The narrow beam of light from the lighthouse rotates at a constant rate of $8\pi$ radians per minute, producing an illuminated spot S that moves along the shoreline. You may assume that the height of the lighthouse can be ignored and that the beam of light lies in the horizontal plane defined by sea level.




When S is 2000 metres from P,

(a) show that the speed of S, correct to three significant figures, is 214 000 metres per minute;

(b) find the acceleration of S.

Using the substitution $x=2\sin\theta$ , show that

$$\int\sqrt{4-x^2}\mathrm{d}x=Ax\sqrt{4-x^2}+B\arcsin\frac{x}{2}+\text{constant},$$

where $A$ and $B$ are constants whose values you are required to find.

---

A particle moves in a straight line in a positive direction from a fixed point O.

The velocity $v$ m s$^{-1}$ , at time $t$ seconds, where $t\geqslant 0$ , satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{-v(1+v^2)}{50}.$$

The particle starts from O with an initial velocity of 10 m s$^{-1}$ .

(a) (i) Express as a definite integral, the time taken for the particle's velocity to decrease from 10 m s$^{-1}$ to 5 m s$^{-1}$ .

(ii) **Hence** calculate the time taken for the particle's velocity to decrease from 10 m s$^{-1}$ to 5 m s$^{-1}$ .

(b) (i) Show that, when $v>0$ , the motion of this particle can also be described by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{-(1+v^2)}{50}$ where $x$ metres is the displacement from O.

(ii) Given that $v=10$ when $x=0$ , solve the differential equation expressing $x$ in terms of $v$.

(iii) **Hence** show that $v=\frac{10-\tan\frac{x}{50}}{1+10\tan\frac{x}{50}}$.

---

Find the gradient of the curve $\mathrm{e}^{xy}+\ln\left(y^2\right)+\mathrm{e}^y=1+\mathrm{e}$ at the point (0, 1) .

---

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is $\theta$ radians.

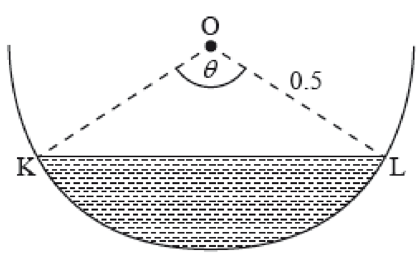



diagram not to scale

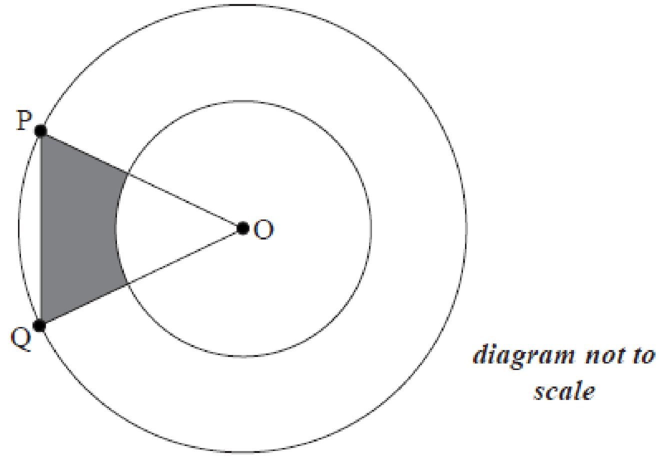
The volume of water is increasing at a constant rate of $0.0008\ \mathrm{m^3s^{-1}}$.

a. Find an expression for the volume of water $V$ $(\mathrm{m}^3)$ in the trough in terms of $\theta$. [3]

b. Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta=\frac{\pi}{3}$. [4]

The diagram below shows two concentric circles with centre O and radii 2 cm and 4 cm.

The points P and Q lie on the larger circle and $\text{P}\hat{\text{O}}\text{Q} = x$ , where $0 < x < \frac{\pi}{2}$ .




(a) Show that the area of the shaded region is $8\sin x - 2x$ .

(b) Find the maximum area of the shaded region.

---

An earth satellite moves in a path that can be described by the curve $72.5x^2 + 71.5y^2 = 1$ where $x = x(t)$ and $y = y(t)$ are in thousands of kilometres and $t$ is time in seconds.

Given that $\frac{\mathrm{d}x}{\mathrm{d}t} = 7.75 \times 10^{-5}$ when $x = 3.2 \times 10^{-3}$, find the possible values of $\frac{\mathrm{d}y}{\mathrm{d}t}$.

Give your answers in standard form.

---

Find the equation of the normal to the curve $y = \frac{\mathrm{e}^x \cos x \ln(x+\mathrm{e})}{(x^{17}+1)^5}$ at the point where $x = 0$.

In your answer give the value of the gradient, of the normal, to three decimal places.

---

Sand is being poured to form a cone of height $h$ cm and base radius $r$ cm. The height remains equal to the base radius at all times. The height of the cone is increasing at a rate of $0.5 \text{ cm min}^{-1}$.

Find the rate at which sand is being poured, in $\text{cm}^3 \text{ min}^{-1}$, when the height is 4 cm.

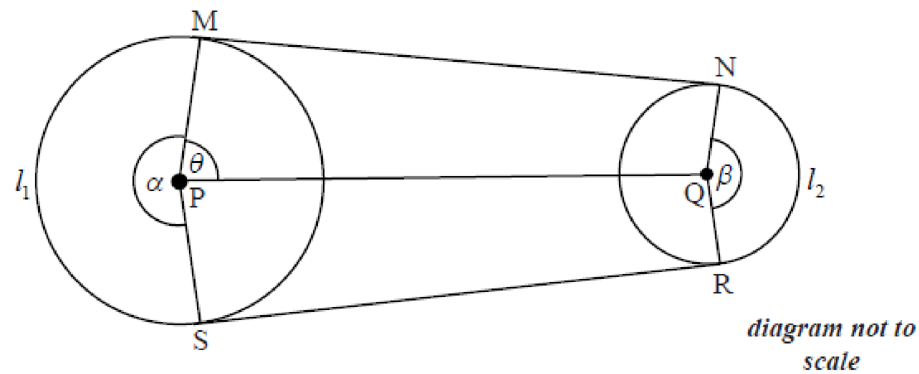
---

By using the substitution $x = \sin t$ , find $\int \frac{x^3}{\sqrt{1-x^2}} \mathrm{d}x$ .

A cone has height $h$ and base radius $r$ . Deduce the formula for the volume of this cone by rotating the triangular region, enclosed by the line $y = h - \frac{h}{r}x$ and the coordinate axes, through $2\pi$ about the $y$-axis.

---

Two non-intersecting circles $C_1$ , containing points M and S , and $C_2$ , containing points N and R, have centres P and Q where $PQ = 50$ . The line segments [MN] and [SR] are common tangents to the circles. The size of the reflex angle MPS is $\alpha$, the size of the obtuse angle NQR is $\beta$ , and the size of the angle MPQ is $\theta$ . The arc length MS is $l_1$ and the arc length NR is $l_2$ . This information is represented in the diagram below.




The radius of $C_1$ is $x$ , where $x \geqslant 10$ and the radius of $C_2$ is 10.

(a) Explain why $x < 40$ .

(b) Show that cosθ = x −10 50.

(c) (i) Find an expression for MN in terms of $x$ .

(ii) Find the value of $x$ that maximises MN.

(d) Find an expression in terms of $x$ for

(i) $\alpha$ ;

(ii) $\beta$ .

(e) The length of the perimeter is given by $l_1 + l_2 + \mathrm{MN} + \mathrm{SR}$.

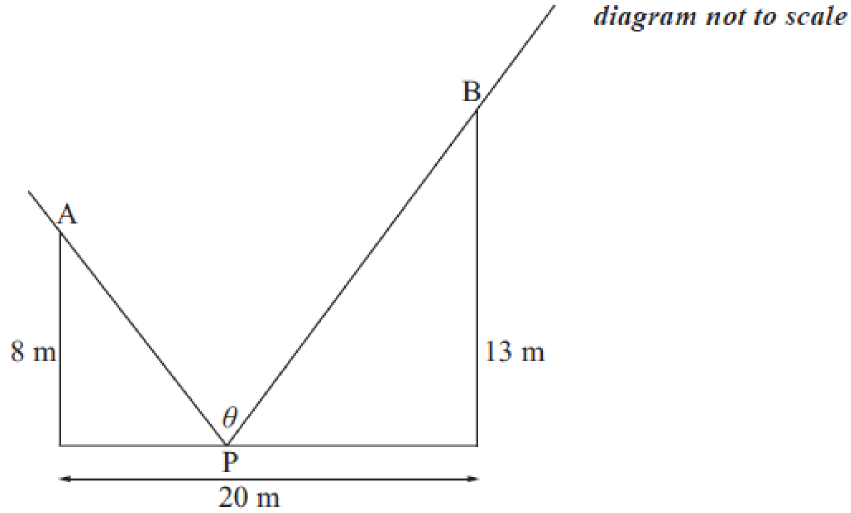
(i) Find an expression, $b(x)$ , for the length of the perimeter in terms of $x$ .

(ii) Find the maximum value of the length of the perimeter.

(iii) Find the value of $x$ that gives a perimeter of length 200.

---

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle $\theta$ where $\theta = \mathrm{A\hat{P}B}$, as shown in the diagram.




a. Find an expression for $\theta$ in terms of $x$, where $x$ is the distance of P from the base of the wall of height 8 m. [2]

b. (i) Calculate the value of $\theta$ when $x = 0$. [2]

(ii) Calculate the value of $\theta$ when $x = 20$.

c. Sketch the graph of $\theta$, for $0 \leqslant x \leqslant 20$. [2]

d. Show that $\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{5(744 - 64x - x^2)}{(x^2+64)(x^2-40x+569)}$. [6]

e. Using the result in part (d), or otherwise, determine the value of $x$ corresponding to the maximum light intensity at P. Give your answer to four significant figures. [3]

f. The point P moves across the street with speed $0.5\ \mathrm{ms}^{-1}$. Determine the rate of change of $\theta$ with respect to time when P is at the midpoint of the street. [4]

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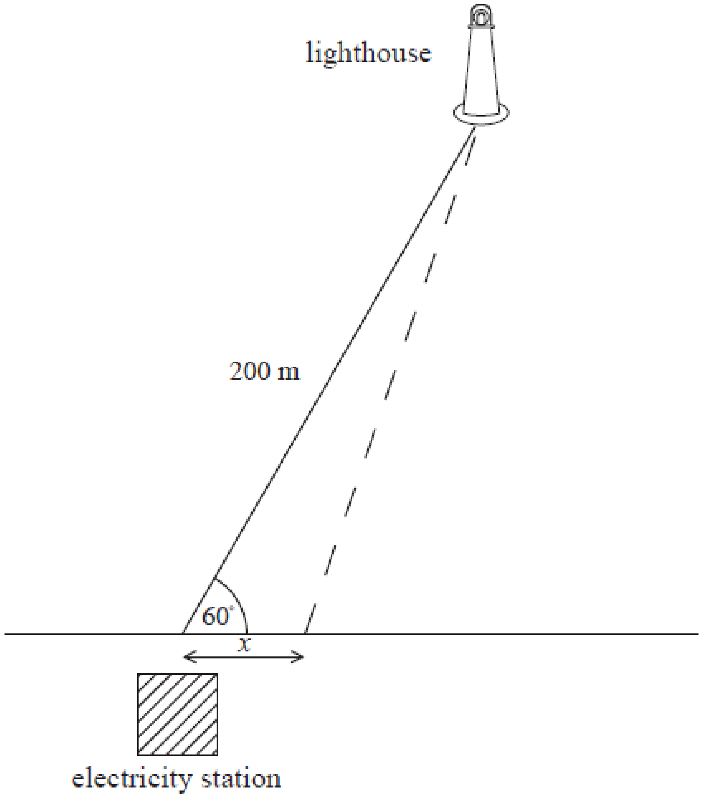
An electricity station is on the edge of a straight coastline. A lighthouse is located in the sea 200 m from the electricity station. The angle between the coastline and the line joining the lighthouse with the electricity station is 60°. A cable needs to be laid connecting the lighthouse to the electricity station. It is decided to lay the cable in a straight line to the coast and then along the coast to the electricity station. The length of cable laid along the coastline is $x$ metres. This information is illustrated in the diagram below.




The cost of laying the cable along the sea bed is US$80 per metre, and the cost of laying it on land is US$20 per metre.

a. Find, in terms of $x$, an expression for the cost of laying the cable. [4]

b. Find the value of $x$, to the nearest metre, such that this cost is minimized. [2]

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a. Sketch the curve $y = \frac{\cos x}{\sqrt{x^2+1}}, \quad -4 \leqslant x \leqslant 4$ showing clearly the coordinates of the $x$-intercepts, any maximum points and any minimum points. [4]

b. Write down the gradient of the curve at $x = 1$ . [1]

c. Find the equation of the normal to the curve at $x = 1$ . [3]

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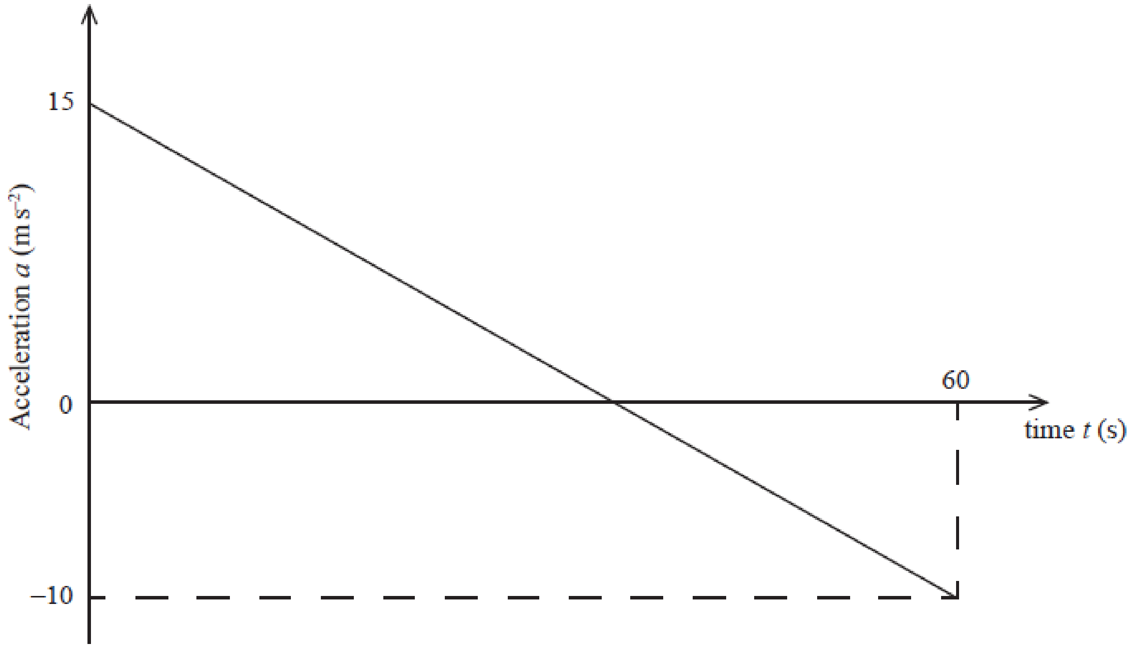
A jet plane travels horizontally along a straight path for one minute, starting at time $t = 0$ , where $t$ is measured in seconds. The acceleration, $a$ , measured in $\text{ms}^{-2}$, of the jet plane is given by the straight line graph below.




a. Find an expression for the acceleration of the jet plane during this time, in terms of $t$ . [1]

b. Given that when $t = 0$ the jet plane is travelling at 125 ms$^{-1}$, find its maximum velocity in ms$^{-1}$ during the minute that follows. [4]

c. Given that the jet plane breaks the sound barrier at 295 ms$^{-1}$, find out for how long the jet plane is travelling greater than this speed. [3]

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A particle, A, is moving along a straight line. The velocity, $v_A$ ms$^{-1}$, of A $t$ seconds after its motion begins is given by

$$v_A = t^3 - 5t^2 + 6t.$$

a. Sketch the graph of $v_A = t^3 - 5t^2 + 6t$ for $t \geqslant 0$, with $v_A$ on the vertical axis and $t$ on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the $t$-axis. [3]

b. Write down the times for which the velocity of the particle is increasing. [2]

c. Write down the times for which the magnitude of the velocity of the particle is increasing. [3]

d. At $t = 0$ the particle is at point O on the line. [3]

Find an expression for the particle's displacement, $x_A$m, from O at time $t$.

e. A second particle, B, moving along the same line, has position $x_B$ m, velocity $v_B$ ms$^{-1}$ and acceleration, $a_B$ ms$^{-2}$, where $a_B = -2v_B$ for $t \geqslant 0$. At $t = 0,\ x_B = 20$ and $v_B = -20$. [4]

Find an expression for $v_B$ in terms of $t$.

f. Find the value of $t$ when the two particles meet. [6]

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The function $f$ is defined by

$$f(x) = (x^3 + 6x^2 + 3x - 10)^{\frac{1}{2}}, \text{ for } x \in D,$$

where $D \subseteq \mathbb{R}$ is the greatest possible domain of *f*.

(a) Find the roots of $f(x) = 0$.

(b) Hence specify the set $D$.

(c) Find the coordinates of the local maximum on the graph $y = f(x)$.

(d) Solve the equation $f(x) = 3$.

(e) Sketch the graph of $|y| = f(x), \text{ for } x \in D$.

(f) Find the area of the region completely enclosed by the graph of $|y| = f(x)$

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The function *f* is defined on the domain [0, 2] by $f(x) = \ln(x+1)\sin(\pi x)$ .

a. Obtain an expression for $f'(x)$ . [3]

b. Sketch the graphs of *f* and $f'$ on the same axes, showing clearly all *x*-intercepts. [4]

c. Find the *x*-coordinates of the two points of inflexion on the graph of *f* . [2]

d. Find the equation of the normal to the graph of *f* where $x = 0.75$ , giving your answer in the form $y = mx + c$ . [3]

e. Consider the points A $(a\ ,\ f(a))$ , B $(b\ ,\ f(b))$ and C $(c\ ,\ f(c))$ where $a$ , $b$ and $c$ $(a < b < c)$ are the solutions of the equation $f(x) = f'(x)$ . Find the area of the triangle ABC. [6]

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Consider the graph of $y = x + \sin(x - 3), \quad -\pi \leqslant x \leqslant \pi.$

a. Sketch the graph, clearly labelling the *x* and *y* intercepts with their values. [3]

b. Find the area of the region bounded by the graph and the *x* and *y* axes. [2]

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Consider the curve with equation $f(x) = \mathrm{e}^{-2x^2}$ for $x < 0$ .

Find the coordinates of the point of inflexion and justify that it is a point of inflexion.

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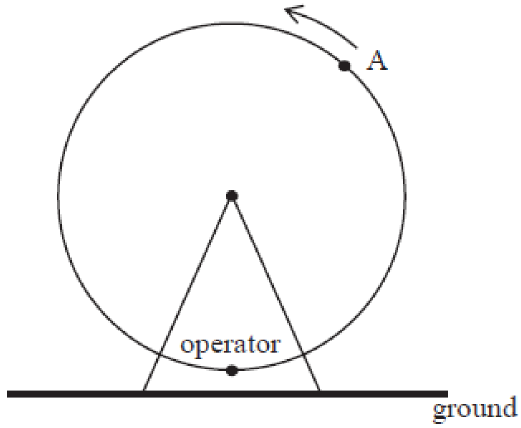
Below is a sketch of a Ferris wheel, an amusement park device carrying passengers around the rim of the wheel.




(a) The circular Ferris wheel has a radius of 10 metres and is revolving at a rate of 3 radians per minute. Determine how fast a passenger on the wheel is going vertically upwards when the passenger is at point A, 6 metres higher than the centre of the wheel, and is rising.

(b) The operator of the Ferris wheel stands directly below the centre such that the bottom of the Ferris wheel is level with his eyeline. As he watches the passenger his line of sight makes an angle $\alpha$ with the horizontal. Find the rate of change of $\alpha$ at point A.

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A particle moves in a straight line with velocity $v$ metres per second. At any time $t$ seconds, $0 \leqslant t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} + v^2 + 1 = 0$ .
It is also given that $v = 1$ when $t = 0$ .

a. Find an expression for $v$ in terms of $t$ . [7]

b. Sketch the graph of $v$ against $t$ , clearly showing the coordinates of any intercepts, and the equations of any asymptotes. [3]

c. (i) Write down the time $T$ at which the velocity is zero. [3]

(ii) Find the distance travelled in the interval $[0, T]$ .

d. Find an expression for $s$ , the displacement, in terms of $t$ , given that $s = 0$ when $t = 0$ . [5]

e. Hence, or otherwise, show that $s = \frac{1}{2}\ln\frac{2}{1+v^2}$. [4]

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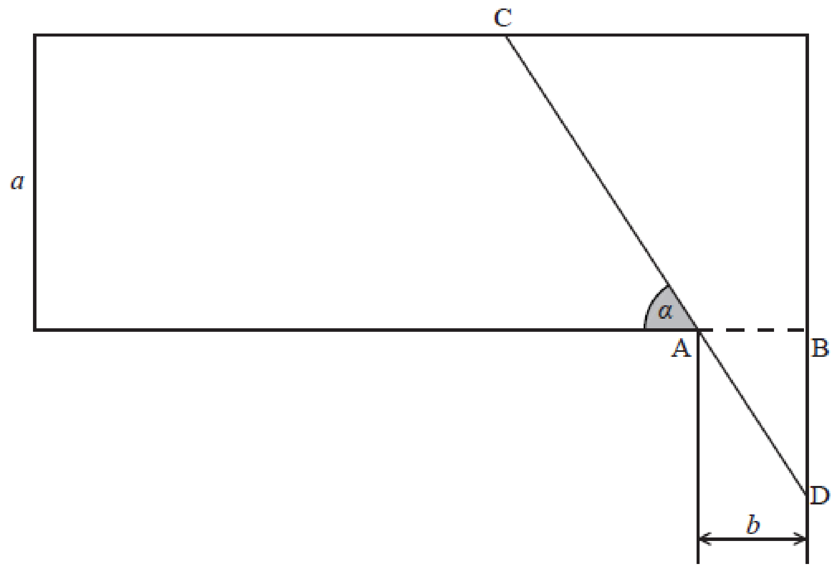
The diagram shows the plan of an art gallery $a$ metres wide. [AB] represents a doorway, leading to an exit corridor $b$ metres wide. In order to remove a painting from the art gallery, CD (denoted by $L$ ) is measured for various values of $\alpha$ , as represented in the diagram.




a. If $\alpha$ is the angle between [CD] and the wall, show that $L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$, $0 < \alpha < \frac{\pi}{2}$. [3]

b. If $a = 5$ and $b = 1$, find the maximum length of a painting that can be removed through this doorway. [4]

c. Let $a = 3k$ and $b = k$. [3]

Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$.

d. Let $a = 3k$ and $b = k$. [6]

Find, in terms of $k$, the maximum length of a painting that can be removed from the gallery through this doorway.

e. Let $a = 3k$ and $b = k$. [2]

Find the minimum value of $k$ if a painting 8 metres long is to be removed through this doorway.

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Find the area of the region enclosed by the curves $y = x^3$ and $x = y^2 - 3$.

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Consider the graphs $y = \mathrm{e}^{-x}$ and $y = \mathrm{e}^{-x}\sin 4x$, for $0 \leqslant x \leqslant \frac{5\pi}{4}$.

(a) On the same set of axes draw, on graph paper, the graphs, for $0 \leqslant x \leqslant \frac{5\pi}{4}$. Use a scale of 1 cm to $\frac{\pi}{8}$ on your $x$-axis and 5 cm to 1 unit on your $y$-axis.

(b) Show that the $x$-intercepts of the graph $y = \mathrm{e}^{-x}\sin 4\mathrm{x}$ are $\frac{n\pi}{4}$, $n = 0, 1, 2, 3, 4, 5$.

(c) Find the $x$-coordinates of the points at which the graph of $y = \mathrm{e}^{-x}\sin 4x$ meets the graph of $y = \mathrm{e}^{-x}$. Give your answers in terms of $\pi$.

(d) (i) Show that when the graph of $y = \mathrm{e}^{-x} \sin 4x$ meets the graph of $y = \mathrm{e}^{-x}$ , their gradients are equal.

(ii) Hence explain why these three meeting points are not local maxima of the graph $y = \mathrm{e}^{-x} \sin 4x$ .

(e) (i) Determine the $y$-coordinates, $y_1$ , $y_2$ and $y_3$, where $y_1 > y_2 > y_3$ , of the local maxima of $y = \mathrm{e}^{-x} \sin 4x$ for $0 \leqslant x \leqslant \frac{5\pi}{4}$ . You do not need to show that they are maximum values, but the values should be simplified.

(ii) Show that $y_1$ , $y_2$ and $y_3$ form a geometric sequence and determine the common ratio $r$ .